

The compositional inverse of a class of bilinear permutation polynomials over finite fields of characteristic 2*

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Abstract

A class of bilinear permutation polynomials over a finite field of characteristic 2 was constructed in a recursive manner recently which involved some other constructions as special cases. We determine the compositional inverses of them based on a direct sum decomposition of the finite field. The result generalizes that in [R.S. Coulter, M. Henderson, The compositional inverse of a class of permutation polynomials over a finite field, Bull. Austral. Math. Soc. 65 (2002) 521-526].

Keywords Permutation polynomial; Bilinear polynomial; Compositional inverse; Direct sum.

1 Introduction

Let \mathbb{F}_q be the finite field with q elements where q is a prime or a prime power, and $\mathbb{F}_q[x]$ be the ring of polynomials over \mathbb{F}_q . For any $f(x) \in \mathbb{F}_q[x]$, it can induce a map from \mathbb{F}_q to itself. $f(x)$ is called a permutation polynomial if the map induced by it is bijective. In fact, we need only to consider polynomials of degree less than q when talking about permutation behavior

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of them. Clearly, under the operation of composition of polynomials and subsequent reduction modulo $(x^q - x)$, the set of all permutation polynomials over \mathbb{F}_q forms a group which is isomorphic to \mathcal{S}_q , the symmetric group on q letters. Hence for any permutation polynomial $f(x) \in \mathbb{F}_q[x]$, there exists a unique polynomial $f^{-1}(x) \in \mathbb{F}_q[x]$ such that $f(f^{-1}(x)) \equiv f^{-1}(f(x)) \equiv x \pmod{(x^q - x)}$. f^{-1} is called the compositional inverse of f (or vice versa).

Discovering new classes of permutation polynomials is an old and important problem due to their applicable value in cryptography, coding theory and combinatorics. However, it is far from easy to do this. There are only a few classes of permutation polynomials known. See [9, 10] for a survey of this topic and [13, 1, 5, 8, 16], for example, for some recent progresses.

Given a class of permutation polynomials, it seems to be an even more difficult problem to find the class of permutation polynomials that represent their compositional inverses. It was noted in [6] that only for permutation linear polynomials, monomials and Dickson polynomials, the compositional inverses could be explicitly determined. To the knowledge of the authors, this list was enlarged in recent years and compositional inverses of the following several other classes of permutation polynomials were determined:

(1) Permutation polynomials of the form $x^r f(x^s)$ over \mathbb{F}_q where $s|(q-1)$. Permutation behavior of such polynomials was studied in [12, 17, 2], and their compositional inverses were obtained in [14].

(2) The linearized polynomials over \mathbb{F}_{q^n} . A polynomial over \mathbb{F}_{q^n} of the shape $L(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ is called a linearized polynomial. It is well known that $L(x)$ is a permutation polynomial if and only if the matrix

$$D_L = \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1}^q & a_0^q & \dots & a_{n-2}^q \\ \vdots & \vdots & & \vdots \\ a_1^{q^{n-1}} & a_2^{q^{n-1}} & \dots & a_0^{q^{n-1}} \end{pmatrix}$$

is non-singular [11]. In [15] the authors found that the compositional inverse of $L(x)$ can be represented by cofactors of elements in the first column of D_L (see [15, Theorem 4.5]).

(3) The bilinear polynomial $x(\text{Tr}(x) + ax)$ over \mathbb{F}_{q^n} where q is even and n is odd, proposed in [3]. Its compositional inverse was determined in [6] which is of a complicated form.

In this paper, we focus on extending the list above. More definitely, we want to replace (3) in the above list with a more general case. A polynomial

over \mathbb{F}_{q^n} is called a DO polynomial if it is of the shape

$$\sum_{0 \leq i, j \leq n-1} a_{ij} x^{q^i + q^j}.$$

A polynomial of the shape $L_1(x)L_2(x)$ for two linearized polynomials over \mathbb{F}_{q^n} is obviously a DO polynomial, which is called a bilinear polynomial in [3]. It was also raised as a problem finding bilinear permutation polynomials in [3], which could be reduced to the problem of finding bilinear permutation polynomials of the shape $xL(x)$ for a linearized polynomial $L(x)$. The special class in (3) above was constructed (see [3, Theorem 5]) and the compositional inverse class was obtained afterwards (see [6, Theorem 1]).

We notice that the class in (3) was generalized in [4] in a recursive manner recently, thus it is a natural question how to generalize its compositional inverse to the compositional inverse of the generalized class of bilinear permutation polynomials. This is not direct since the method in [6] to obtain compositional inverse is a “guess and determine” one: verifying the result after guessing it based on some experimental evidences. However, after further studying properties of such permutation polynomials, we can overcome this difficulty. The main idea of our method is to decompose the finite field into a direct sum of two subspaces, and represent the map induced by a univariate permutation polynomial $f(x)$ by a bivariate permutation polynomial system $\mathbf{f}(y, z) = (f_1(y, z), f_2(y, z))$. In the case $f(x)$ is a bilinear polynomial we consider, the corresponding bivariate polynomial system $\mathbf{f}(y, z)$ is of a triangular form: $f_1(y, z)$ is independent of the variable z . We can get the inverse polynomial system after overcoming the difficulty of determining the inverse of a permutation induced by a linearized polynomial on a component of the direct sum decomposition of the finite field. To summarize, we transform the problem of computing inverse of a non-linear map on the finite field to the problem of computing inverse of a linear map on a subspace of it, which seems much easier to solve.

The rest of the paper is organized as follows. In Section 2 we recall some constructions of bilinear permutation polynomials and determine their compositional inverses. In Section 3 we explain our method to obtain the results. Concluding remarks are given in Section 4.

2 Bilinear permutations and their compositional inverses

We denote the trace map from \mathbb{F}_{q^n} to \mathbb{F}_q by $\text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}$ or Tr for simplicity when it will not cause confusion, that is

$$\text{Tr}(x) = \sum_{i=0}^{n-1} x^{q^i}, \quad x \in \mathbb{F}_{q^n}.$$

Throughout the rest of the paper we only consider finite fields of characteristic 2.

We firstly recall the construction of a class of bilinear permutation polynomials proposed in [3].

Theorem 2.1 ([3]). *Let q be even and n be odd. Then the polynomial*

$$f(x) = x (\text{Tr}(x) + ax)$$

is a permutation polynomial over \mathbb{F}_{q^n} for all $a \in \mathbb{F}_q \setminus \{0, 1\}$.

In [4] Laigle-Chapuy generalized this construction in a recursive manner.

Theorem 2.2 ([4]). *Let q be even and n be odd. Assume $xL(x)$ is a bilinear permutation polynomial over \mathbb{F}_q for a linearized polynomial $L(x) \in \mathbb{F}_q[x]$. Then the polynomial*

$$F(x) = x (L(\text{Tr}(x)) + a\text{Tr}(x) + ax)$$

is a bilinear permutation polynomial over \mathbb{F}_{q^n} for any $a \in \mathbb{F}_q^$.*

Note that the polynomial in Theorem 2.1 can be derived from Theorem 2.2 by setting $L(x) = x$. In the following we will propose the compositional inverse of $F(x)$ given in Theorem 2.2. Firstly we remark that in representing maps from \mathbb{F}_{q^n} to itself by polynomials over \mathbb{F}_{q^n} , we sometimes distinguish $\frac{1}{x}$ with x^{q^n-2} . For example, we use $\frac{1}{\text{Tr}(x)}$ to represent $\text{Tr}(x)^{q^n-2} = \text{Tr}(x)^{q-2}$. Besides, we sometimes use $x^{1/2}$ instead of $x^{q^n/2}$.

Theorem 2.3. *Use the same notations as in Theorem 2.2 and let $q = 2^m$ for a positive integer m . Assume the compositional inverse of $xL(x)$ is $g(x) \in \mathbb{F}_q[x]$. Then*

$$\begin{aligned}
F^{-1}(x) &= a^{2^{m-1}-1}x^{2^{nm}-1} + \left(g(\text{Tr}(x)) + a^{2^{m-1}-1} \sum_{k=1}^{\frac{n-1}{2}} x^{2^{(2k-1)m-1}} \right) \\
&\quad \left(\frac{\text{Tr}(x)}{g(\text{Tr}(x))} + ag(\text{Tr}(x)) \right)^{q-1} \\
&\quad + \sum_{j=0}^{m-2} a^{2^j-1} \left(\frac{\text{Tr}(x)}{g(\text{Tr}(x))} + ag(\text{Tr}(x)) \right)^{2^m-2^{j+1}} \left(\sum_{k=0}^{\frac{n-1}{2}} x^{q^{2k}} \right)^{2^j}.
\end{aligned}$$

Proof. We proceed by directly verifying $F^{-1}(F(x)) = x$ under subsequent reduction modulo $(x^{q^n} - x)$. Note that for any $x \in \mathbb{F}_{q^n}$,

$$F^{-1}(x) = \left(\frac{x}{a} \right)^{1/2}$$

when $\frac{\text{Tr}(x)}{g(\text{Tr}(x))} + ag(\text{Tr}(x)) = 0$, and

$$\begin{aligned}
F^{-1}(x) &= g(\text{Tr}(x)) + \sum_{j=0}^{m-1} \frac{a^{2^j-1}}{\left(\frac{\text{Tr}(x)}{g(\text{Tr}(x))} + ag(\text{Tr}(x)) \right)^{2^{j+1}-1}} \left(\sum_{k=0}^{\frac{n-1}{2}} x^{q^{2k}} \right)^{2^j} \\
&= g(\text{Tr}(x)) + \sum_{j=0}^{m-1} \frac{a^{2^j-1}}{\left(\frac{\text{Tr}(x)}{g(\text{Tr}(x))} + ag(\text{Tr}(x)) \right)^{2^{j+1}-1}} \sum_{k=0}^{\frac{n-1}{2}} x^{2^{2km+j}}
\end{aligned}$$

otherwise.

Firstly, it is obvious that

$$\begin{aligned}
\text{Tr}(F(x)) &= \text{Tr}(xL(\text{Tr}(x)) + ax\text{Tr}(x) + ax^2) \\
&= \text{Tr}(x)L(\text{Tr}(x)) + a\text{Tr}(x)^2 + a\text{Tr}(x^2) \\
&= \text{Tr}(x)L(\text{Tr}(x))
\end{aligned}$$

as $\text{Tr}(1) = 1$ and $\text{Tr}(x^2) = \text{Tr}(x)^2$. Thus

$$g(\text{Tr}(F(x))) = g(\text{Tr}(x)L(\text{Tr}(x))) = \text{Tr}(x)$$

since $g(xL(x)) = x$. Besides, for any $x \in \mathbb{F}_{q^n}$, it is clear that $L(\text{Tr}(x)) + a\text{Tr}(x) = 0$ if and only if $\frac{\text{Tr}(F(x))}{g(\text{Tr}(F(x)))} + ag(\text{Tr}(F(x))) = 0$ since

$$\frac{\text{Tr}(F(x))}{g(\text{Tr}(F(x)))} + ag(\text{Tr}(F(x))) = \frac{\text{Tr}(x)L(\text{Tr}(x))}{\text{Tr}(x)} + a\text{Tr}(x).$$

When $L(\text{Tr}(x)) + a\text{Tr}(x) = 0$, we have $F(x) = ax^2$ and hence

$$F^{-1}(F(x)) = \left(\frac{x}{a}\right)^{1/2} \circ (ax^2) = x;$$

When $L(\text{Tr}(x)) + a\text{Tr}(x) \neq 0$, we have

$$\begin{aligned} F^{-1}(F(x)) &= \text{Tr}(x) + \sum_{j=0}^{m-1} \frac{a^{2^j-1}}{\left(\frac{\text{Tr}(x)L(\text{Tr}(x))}{\text{Tr}(x)} + a\text{Tr}(x)\right)^{2^{j+1}-1}} \cdot \\ &\quad \sum_{k=0}^{\frac{n-1}{2}} (xL(\text{Tr}(x)) + ax\text{Tr}(x) + ax^2)^{2^{2km+j}} \\ &= \text{Tr}(x) + \sum_{j=0}^{m-1} \frac{a^{2^j-1}}{(L(\text{Tr}(x)) + a\text{Tr}(x))^{2^{j+1}-1}} \cdot \\ &\quad \left[(L(\text{Tr}(x)) + a\text{Tr}(x))^{2^j} \sum_{k=0}^{\frac{n-1}{2}} x^{2^{2km+j}} + a^{2^j} \sum_{k=0}^{\frac{n-1}{2}} x^{2^{2km+j+1}} \right] \\ &= \text{Tr}(x) + \sum_{j=0}^{m-1} \frac{a^{2^j-1}}{(L(\text{Tr}(x)) + a\text{Tr}(x))^{2^j-1}} \sum_{k=0}^{\frac{n-1}{2}} x^{2^{2km+j}} \\ &\quad + \sum_{j=0}^{m-1} \frac{a^{2^{j+1}-1}}{(L(\text{Tr}(x)) + a\text{Tr}(x))^{2^{j+1}-1}} \sum_{k=0}^{\frac{n-1}{2}} x^{2^{2km+j+1}} \\ &= \text{Tr}(x) + \sum_{k=0}^{\frac{n-1}{2}} x^{2^{2km}} + \sum_{k=0}^{\frac{n-1}{2}} x^{2^{(2k+1)m}} \\ &= \text{Tr}(x) + \sum_{k=0}^{n-1} x^{q^k} + x^{q^n} \end{aligned}$$

$$= x.$$

To summarize, we have $F^{-1}(F(x)) = x$ for any $x \in \mathbb{F}_{q^n}$. \square

Corollary 2.4. *Let $q = 2^m$ and n be odd. Let $f(x)$ be the bilinear permutation polynomial defined in Theorem 2.1. Then*

$$\begin{aligned} f^{-1}(x) &= a^{2^{m-1}-1} x^{2^{nm-1}} + (1+a)^{2^{m-1}-1} \text{Tr}(x)^{2^{m-1}} \\ &\quad + a^{2^{m-1}-1} \text{Tr}(x)^{2^{m-1}(2^m-1)} \sum_{k=1}^{\frac{n-1}{2}} x^{2^{(2k-1)m-1}} \\ &\quad + \sum_{j=0}^{m-2} a^{2^j-1} (1+a)^{2^{m-1}+2^j-1} \text{Tr}(x)^{2^{m-1}-2^j} \left(\sum_{k=0}^{\frac{n-1}{2}} x^{q^{2k}} \right)^{2^j}. \end{aligned}$$

Proof. We let $L(x) = x$ and replace a with $\frac{a}{1+a}$ in Theorem 2.2. Then

$$\begin{aligned} F(x) &= x \left(\frac{1}{1+a} \text{Tr}(x) + \frac{a}{1+a} x \right) \\ &= \frac{1}{1+a} x (\text{Tr}(x) + ax) \\ &= \frac{1}{1+a} f(x). \end{aligned}$$

Hence $f^{-1}(x) = F^{-1}\left(\frac{x}{1+a}\right)$. Then the result is obtained from Theorem 2.3 noting that $(xL(x))^{-1} = x^{1/2}$. \square

Remark 2.5. *[6, Theorem 1] can be got from Corollary 2.4 by replacing a by $1 + \alpha$ for $\alpha \in \mathbb{F}_q \setminus \{0, 1\}$. Though the compositional inverse deduced from Corollary 2.4 is not totally the same with that in [6, Theorem 1] in representation, they are indeed the same polynomial after collections of terms.*

Recently, Dempwolff and Müller constructed a new class of bilinear permutation polynomials in [7] using trace maps over a tower of finite fields in constructing translation planes of even order.

Theorem 2.6 ([7]). *Let d_1, d_2, \dots, d_h, n be all positive integers satisfying that $d_1|d_2|\dots|d_h|n$ and $\frac{n}{d_1}$ is odd. Let $c_i \in \mathbb{F}_{2^{d_i}}^*$, $1 \leq i \leq h$, such that*

$\sum_{j=1}^i c_j \neq 0$ for all i . Choose $1 \leq l < d_1$ with $\gcd(2^{d_1} - 1, 2^l + 1) = 1$ and $c_0 \in \mathbb{F}_{2^{d_1}}^*$. Set

$$L_{h+1}(x) = \left(\sum_{i=1}^h c_i \right) x + \sum_{i=1}^h c_i T_{n:d_i}(x) + c_0 T_{n:d_1}(x)^{2^l},$$

where $T_{n:d_i}$ denotes the trace map from \mathbb{F}_{2^n} to $\mathbb{F}_{2^{d_i}}$, $1 \leq i \leq h$. Then $F_{h+1}(x) = xL_{h+1}(x)$ is a bilinear permutation polynomial over \mathbb{F}_{2^n} .

Actually, we find that the polynomial $F_{h+1}(x)$ in Theorem 2.6 can be obtained from Theorem 2.2 recursively (though maybe it was discovered independently) in the following manner. Set

$$L_1(x) = c_0 x^{2^l} \in \mathbb{F}_{2^{d_1}}[x]$$

and

$$L_i(x) = L_{i-1}(T_{d_i:d_{i-1}}(x)) + \left(\sum_{j=1}^{i-1} c_j \right) T_{d_i:d_{i-1}}(x) + \left(\sum_{j=1}^{i-1} c_j \right) x \in \mathbb{F}_{2^{d_i}}[x]$$

for $2 \leq i \leq h$. Finally we set

$$L_{h+1}(x) = L_h(T_{n:d_h}(x)) + \left(\sum_{j=1}^h c_j \right) T_{n:d_h}(x) + \left(\sum_{j=1}^h c_j \right) x \in \mathbb{F}_{2^n}[x]$$

From the transitivity of the trace map, it is easy to verify that $L_{h+1}(x)$ is just the one defined in Theorem 2.6. By Theorem 2.2, we directly obtain that $F_i(x) = xL_i(x)$ is a permutation polynomial over $\mathbb{F}_{2^{d_i}}$ for $2 \leq i \leq h$ and $F_{h+1}(x)$ is a permutation polynomial over \mathbb{F}_{2^n} , since $F_1(x) = xL_1(x)$ is a permutation polynomial over $\mathbb{F}_{2^{d_1}}$ due to $\gcd(2^{d_1} - 1, 2^l + 1) = 1$. Hence $F_{h+1}^{-1}(x)$ can be derived from Theorem 2.3 inductively.

Corollary 2.7. *Use notations the same as in Theorem 2.6 and the discussions after it, and let $d_{h+1} = n$. Assume $u(2^l + 1) \equiv 1 \pmod{(2^{d_1} - 1)}$ and $a_i = \sum_{j=1}^{i-1} c_j$, $2 \leq i \leq h + 1$. Then*

$$F_1^{-1}(x) = \left(\frac{x}{c_0} \right)^u,$$

and for $2 \leq i \leq h+1$,

$$\begin{aligned}
F_i^{-1}(x) &= a_i^{2^{d_{i-1}-1}-1} x^{2^{d_i-1}} \\
&+ \left(F_{i-1}^{-1}(T_{d_i:d_{i-1}}(x)) + a_i^{2^{d_{i-1}-1}-1} \sum_{k=1}^{\frac{d_i/d_{i-1}-1}{2}} x^{2^{(2k-1)d_{i-1}-1}} \right) \\
&\quad \left(\frac{T_{d_i:d_{i-1}}(x)}{F_{i-1}^{-1}(T_{d_i:d_{i-1}}(x))} + a_i T_{d_i:d_{i-1}}(x) \right)^{2^{d_{i-1}-1}-1} \\
&+ \sum_{j=0}^{d_{i-1}-2} a_i^{2^j-1} \left(\frac{T_{d_i:d_{i-1}}(x)}{F_{i-1}^{-1}(T_{d_i:d_{i-1}}(x))} + a_i T_{d_i:d_{i-1}}(x) \right)^{2^{d_{i-1}-2j+1}} \sum_{k=0}^{\frac{d_i/d_{i-1}-1}{2}} x^{2^{2kd_{i-1}+j}}.
\end{aligned}$$

3 The method to obtain Theorem 2.3

The method we get Theorem 2.3 is not a “guess and determine” one, as a matter of fact. In this section we describe it in detail.

Let q be even and n be odd. The for any $c \in \mathbb{F}_q$, $\text{Tr}(c) = c$, which implies that $\mathbb{F}_q \cap \ker \text{Tr} = \{0\}$, where $\ker \text{Tr}$ is the kernel of the trace map from \mathbb{F}_{q^n} to \mathbb{F}_q . Since \mathbb{F}_q and $\ker \text{Tr}$ are 1-dimensional and $(n-1)$ -dimensional vector spaces over \mathbb{F}_q , respectively, the following lemma is straightforward.

Lemma 3.1. *Let q be even and n be odd. Then*

$$\mathbb{F}_{q^n} \cong \mathbb{F}_q \oplus \ker \text{Tr}.$$

Remark 3.2. *The map to establish the isomorphism in Lemma 3.1 is*

$$\begin{aligned}
\phi : \mathbb{F}_{q^n} &\longrightarrow \mathbb{F}_q \oplus \ker \text{Tr} \\
x &\longmapsto (\text{Tr}(x), x + \text{Tr}(x)),
\end{aligned}$$

and for $(y, z) \in \mathbb{F}_q \oplus \ker \text{Tr}$, $\phi^{-1}((y, z)) = y + z$.

Now we consider the graph

$$\begin{array}{ccc}
\mathbb{F}_{q^n} & \xrightarrow{F(x)} & \mathbb{F}_{q^n} \\
\downarrow \phi & & \downarrow \phi \\
\mathbb{F}_q \oplus \ker \text{Tr} & \xrightarrow{\mathbf{F}(y, z)} & \mathbb{F}_q \oplus \ker \text{Tr}
\end{array}$$

where $F(x)$ is the bilinear permutation polynomial defined in Theorem 2.2, and $\mathbf{F}(y, z)$ is a bivariate polynomial system the map induced by which can make the graph commutative. Let $y = \text{Tr}(x)$ and $z = x + \text{Tr}(x)$. Since

$$\text{Tr}(F(x)) = \text{Tr}(x)L(\text{Tr}(x)) = yL(y)$$

and

$$\begin{aligned}
F(x) + \text{Tr}(F(x)) &= (y + z)L(y) + a(y + z)y + a(y + z)^2 + yL(y) \\
&= az^2 + (L(y) + ay)z,
\end{aligned}$$

we have

$$\mathbf{F}(y, z) = (yL(y), az^2 + (L(y) + ay)z).$$

$yL(y)$ is a permutation polynomial over \mathbb{F}_q as stated, thus $az^2 + (L(y) + ay)z$ can induce a permutation of $\ker \text{Tr}$ for any $y \in \mathbb{F}_q$ since $\mathbf{F}(y, z)$ can induce a permutation of $\mathbb{F}_q \oplus \ker \text{Tr}$. In fact, $\mathbf{F}(y, z)$ is a bivariate polynomial system of the so-called triangular form, so the inverse polynomial system can be obtained in case we can get the inverse of the permutation of $\ker \text{Tr}$.

Lemma 3.3. *Let $q = 2^m$ and n be odd. Let $P_c(x) = x^2 + cx$ for any $c \in \mathbb{F}_q^*$. Then $P_c(x)$ can induce a permutation of $\ker \text{Tr}$ and the polynomial that can induce its inverse map is*

$$P_c^{-1}(x) = \sum_{j=0}^{m-1} c^{-(2^{j+1}-1)} \left(\sum_{k=0}^{\frac{n-1}{2}} x^{q^{2k}} \right)^{2^j}.$$

(We call $P_c(x)$ a permutation polynomial over $\ker \text{Tr}$ and $P_c^{-1}(x)$ its compositional inverse.)

Proof. Obviously, for any $x \in \ker \text{Tr}$, $P_c(x) \in \ker \text{Tr}$. Furthermore, $P_c(x)$ induces an \mathbb{F}_2 -linear transformation of the vector space $\ker \text{Tr}$, the kernel of which is $\{0\}$ since $P_c(x) = 0$ implies $x = 0$ or $x = c$ but $c \notin \ker \text{Tr}$. Hence the linear transformation induced by $P_c(x)$ is invertible. For any $x \in \ker \text{Tr}$,

$$\begin{aligned}
P_c^{-1}(P_c(x)) &= \sum_{j=0}^{m-1} c^{-(2^{j+1}-1)} \sum_{k=0}^{\frac{n-1}{2}} (x^2 + cx)^{2^{2km+j}} \\
&= \sum_{j=0}^{m-1} c^{-(2^{j+1}-1)} \sum_{k=0}^{\frac{n-1}{2}} x^{2^{2km+j+1}} + \sum_{j=0}^{m-1} c^{-(2^j-1)} \sum_{k=0}^{\frac{n-1}{2}} x^{2^{2km+j}} \\
&= \sum_{k=0}^{\frac{n-1}{2}} x^{2^{(2k+1)m}} + \sum_{k=0}^{\frac{n-1}{2}} x^{2^{2km}} \\
&= \text{Tr}(x) + x \\
&= x.
\end{aligned}$$

Hence $P_c^{-1}(x)$ is just the compositional inverse of $P_c(x)$ over $\ker \text{Tr}$. \square

Remark 3.4. In fact, $P_c^{-1}(x)$ in Lemma 3.3 is not got fully by guessing and determining. Since $P_c(x)$ is a linearized permutation polynomial over $\ker \text{Tr}$, we can assume $P_c^{-1}(x) = \sum_{i=0}^{mn-1} d_i x^{2^i}$. However, we cannot expect to get $P_c^{-1}(P_c(x)) = x$ over \mathbb{F}_{q^n} since $P_c(x)$ is not a permutation polynomial over \mathbb{F}_{q^n} . Hence we expect to get $P_c^{-1}(P_c(x)) = x + \text{Tr}(x) = x^{2^m} + x^{2^{2m}} + \dots + x^{2^{(n-1)m}}$ as we will be limited to $\ker \text{Tr}$, which leads to a system of equations

$$\begin{cases} d_{i-1}^2 + cd_i = 0 & \text{if } i \neq km \\ d_{i-1}^2 + cd_i = 1 & \text{if } i = km \end{cases}, \quad 0 \leq i \leq mn-1, \quad 1 \leq k \leq n-1.$$

Solving this system we obtain $d_i = c^{-(2^{j+1}-1)}$ when k is even and $d_i = 0$ when k is odd for $i = km + j$, $0 \leq j \leq m-1$, $0 \leq k \leq n-1$.

Let $\mathbf{F}^{-1}(y, z)$ be the bivariate polynomial system such that for any $(y, z) \in \mathbb{F}_q \oplus \ker \text{Tr}$, $\mathbf{F}^{-1} \circ \mathbf{F}(y, z) = (y, z)$. Now for $(Y, Z) \in \mathbb{F}_q \oplus \ker \text{Tr}$, we assume

$$\begin{cases} yL(y) = Y \\ az^2 + (L(y) + ay)z = Z. \end{cases}$$

Let $g(x) \in \mathbb{F}_q[x]$ be the compositional inverse of the permutation polynomial $xL(x)$ over \mathbb{F}_q . If $L(g(Y)) + ag(Y) = 0$, i.e. $\frac{Y}{g(Y)} + ag(Y) = 0$ (since $g(Y)L(g(Y)) = Y$), we get

$$\begin{cases} y = g(Y) \\ z = \left(\frac{Z}{a}\right)^{1/2}. \end{cases}$$

This is equivalent to say

$$\mathbf{F}^{-1}(Y, Z) = \left(g(Y), \left(\frac{Z}{a}\right)^{1/2}\right) = \left(\left(\frac{Y}{a}\right)^{1/2}, \left(\frac{Z}{a}\right)^{1/2}\right);$$

If $L(g(Y)) + ag(Y) \neq 0$, we get

$$\begin{cases} y = g(Y) \\ z = P_{L(y)/a+y}^{-1}\left(\frac{Z}{a}\right) = P_{Y/(ag(Y))+g(Y)}^{-1}\left(\frac{Z}{a}\right), \end{cases}$$

where $P_c(x)$ is the polynomial defined in Lemma 3.3. By Lemma 3.3 we have

$$\begin{aligned} z &= \sum_{j=0}^{m-1} \left(\frac{Y}{ag(Y)} + g(Y)\right)^{-(2^{j+1}-1)} \sum_{k=0}^{\frac{n-1}{2}} \left(\frac{Z}{a}\right)^{2^{2km+j}} \\ &= \sum_{j=0}^{m-1} \frac{a^{2^j-1}}{\left(\frac{Y}{g(Y)} + ag(Y)\right)^{2^{j+1}-1}} \sum_{k=0}^{\frac{n-1}{2}} Z^{2^{2km+j}}. \end{aligned}$$

Hence

$$\mathbf{F}^{-1}(Y, Z) = \left(g(Y), \sum_{j=0}^{m-1} \frac{a^{2^j-1}}{\left(\frac{Y}{g(Y)} + ag(Y)\right)^{2^{j+1}-1}} \sum_{k=0}^{\frac{n-1}{2}} Z^{2^{2km+j}}\right).$$

Now we go back to \mathbb{F}_{q^n} by setting $X = Y + Z$ where $Y = \text{Tr}(X)$ and $Z = X + \text{Tr}(X)$. We get when $\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X)) = 0$,

$$F^{-1}(X) = \left(\frac{\text{Tr}(X)}{a}\right)^{1/2} + \left(\frac{X + \text{Tr}(X)}{a}\right)^{1/2} = \left(\frac{X}{a}\right)^{1/2},$$

and when $\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X)) \neq 0$,

$$\begin{aligned}
F^{-1}(X) &= g(\text{Tr}(X)) + \sum_{j=0}^{m-1} \frac{a^{2^j-1}}{\left(\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X))\right)^{2^{j+1}-1}} \sum_{k=0}^{\frac{n-1}{2}} (X + \text{Tr}(X))^{2^{2km+j}} \\
&= g(\text{Tr}(X)) + \sum_{j=0}^{m-1} \frac{a^{2^j-1}}{\left(\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X))\right)^{2^{j+1}-1}} \sum_{k=0}^{\frac{n-1}{2}} X^{2^{2km+j}} \\
&\quad + \frac{n+1}{2} \sum_{j=0}^{m-1} \frac{a^{2^j-1} \text{Tr}(X)^{2^j}}{\left(\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X))\right)^{2^{j+1}-1}} \\
&= g(\text{Tr}(X)) + \sum_{j=0}^{m-1} \frac{a^{2^j-1}}{\left(\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X))\right)^{2^{j+1}-1}} \left(\sum_{k=0}^{\frac{n-1}{2}} X^{q^{2k}} \right)^{2^j}
\end{aligned}$$

since

$$\begin{aligned}
&\sum_{j=0}^{m-1} \frac{a^{2^j-1} \text{Tr}(X)^{2^j}}{\left(\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X))\right)^{2^{j+1}-1}} \\
&= \frac{\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X))}{a} \cdot \sum_{j=0}^{m-1} \frac{1}{\frac{\text{Tr}(X)^{2^j}}{a^{2^j} g(\text{Tr}(X))^{2^{j+1}}} + \frac{a^{2^j} g(\text{Tr}(X))^{2^{j+1}}}{\text{Tr}(X)^{2^j}}} \\
&= \frac{\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X))}{a} \cdot \text{Tr}_{\mathbb{F}_{2^m}/\mathbb{F}_2} \left(\frac{1}{\frac{\text{Tr}(X)}{ag(\text{Tr}(X))^2} + \frac{ag(\text{Tr}(X))^2}{\text{Tr}(X)}} \right) \\
&= 0
\end{aligned}$$

using the following lemma.

Lemma 3.5. *Let r be a positive integer. Then for any $e \in \mathbb{F}_{2^r}$,*

$$\text{Tr}_{\mathbb{F}_{2^r}/\mathbb{F}_2} \left(\frac{1}{e + e^{-1}} \right) = 0.$$

(Recall that we distinguish $\frac{1}{0}$ and 0^{-1} with $0^{2^r-2} = 0$.)

Proof. Since

$$\frac{1}{e+e^{-1}} = \frac{e}{e^2+1} = \frac{e+1+1}{(e+1)^2} = \frac{1}{e+1} + \frac{1}{(e+1)^2},$$

the result is straightforward to get. \square

Finally $F^{-1}(X)$ can be derived by interpolation, that is

$$\begin{aligned} F^{-1}(X) &= \left(\frac{X}{a}\right)^{1/2} \left[1 + \left(\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X)) \right)^{q-1} \right] \\ &\quad + \left[g(\text{Tr}(X)) + \sum_{j=0}^{m-1} \frac{a^{2^j-1}}{\left(\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X)) \right)^{2^{j+1}-1}} \left(\sum_{k=0}^{\frac{n-1}{2}} X^{q^{2k}} \right)^{2^j} \right] \\ &\quad \left(\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X)) \right)^{q-1} \\ &= \left(\frac{X}{a}\right)^{1/2} + \left[\left(\frac{X}{a}\right)^{1/2} + g(\text{Tr}(X)) + a^{2^{m-1}-1} \left(\sum_{k=0}^{\frac{n-1}{2}} X^{q^{2k}} \right)^{2^{m-1}} \right] \\ &\quad \left(\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X)) \right)^{q-1} \\ &\quad + \sum_{j=0}^{m-2} a^{2^j-1} \left(\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X)) \right)^{2^m-2^{j+1}} \left(\sum_{k=0}^{\frac{n-1}{2}} X^{q^{2k}} \right)^{2^j} \\ &= a^{2^{m-1}-1} X^{1/2} + \left(g(\text{Tr}(X)) + a^{2^{m-1}-1} \sum_{k=1}^{\frac{n-1}{2}} X^{2^{(2k-1)m-1}} \right) \\ &\quad \left(\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X)) \right)^{q-1} \\ &\quad + \sum_{j=0}^{m-2} a^{2^j-1} \left(\frac{\text{Tr}(X)}{g(\text{Tr}(X))} + ag(\text{Tr}(X)) \right)^{2^m-2^{j+1}} \left(\sum_{k=0}^{\frac{n-1}{2}} X^{q^{2k}} \right)^{2^j}. \end{aligned}$$

Hence Theorem 2.3 is obtained.

4 Concluding remarks

In this paper, we derive the compositional inverse of a class of bilinear permutation polynomials over a finite field of characteristic 2, which can serve as a new class of permutation polynomials over finite fields. The main observation we make is the special structure of the maps induced by this class of permutation polynomials: they can be represented by a class of bivariate polynomial systems which are of triangular shapes after decomposing the finite field into a direct sum of a subfield and the kernel space of the trace map. In fact, the class of bilinear polynomials in Theorem 2.6 can also be represented by multivariate polynomial systems of triangular shapes considering the decomposition

$$\mathbb{F}_{2^n} \cong \mathbb{F}_{2^{d_1}} \oplus \ker T_{d_2:d_1} \oplus \ker T_{d_3:d_2} \oplus \cdots \oplus \ker T_{n:d_h}.$$

Indeed, the corresponding polynomial systems are of the form

$$\begin{pmatrix} x_0 L_1(x_0), \\ x_1 [L_1(x_0) + c_1 x_0] + c_1 x_1^2, \\ x_2 [L_2(x_0 + x_1) + (c_1 + c_2)(x_0 + x_1)] + (c_1 + c_2) x_2^2, \\ \vdots \\ x_i \left[L_i \left(\sum_{j=0}^{i-1} x_j \right) + \left(\sum_{j=1}^i c_j \right) \left(\sum_{j=0}^{i-1} x_j \right) \right] + \left(\sum_{j=1}^i c_j \right) x_i^2, \\ \vdots \\ x_h \left[L_h \left(\sum_{j=0}^{h-1} x_j \right) + \left(\sum_{j=1}^h c_j \right) \left(\sum_{j=0}^{h-1} x_j \right) \right] + \left(\sum_{j=1}^h c_j \right) x_h^2 \end{pmatrix}.$$

Inverse polynomial systems of them can also be computed inductively using Lemma 3.3, which will lead to Corollary 2.7 after lifted back to univariate polynomials.

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